

Explicit Formulae for L -values in Finite Characteristic

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Abstract

We give an elementary treatment making explicit Pellarin's theorems regarding his new twist on Goss' characteristic p valued L -series. Our method extends to show that a slightly more general form of these series also turn out to be rational functions when multiplied by a certain canonical factor. We further give a closed form for Pellarin's L -series at 1, and we reproduce his proof using a certain operator formalism inspired by Anderson's t -motives. Along the way we prove what we call *The Fundamental Relation* relating these more general series to the Carlitz logarithm. This relation will play a key role in our exposition of Pellarin's proof, and we shall also use it to give closed form formulas for a certain family of Dirichlet L -series at 1.

1 Introduction

In all that follows below, let $q = p^{m_0}$ be a fixed power of the *odd* prime p . Let $A = \mathbb{F}_q[\theta]$ and $K = \mathbb{F}_q(\theta)$. Let $|\cdot|_\infty = |\cdot|$ be the non-archimedean absolute value on K which assigns $|\theta| = q$. Let K_∞ be the completion of K with respect to this absolute value, and let \mathbb{C}_∞ be the completion of an algebraic closure of K_∞ equipped with the canonical extension of $|\cdot|_\infty$. We let A_+ and $A_+(d)$ be the collection of all monic polynomials and the collection of all monic polynomials of degree d respectively. We let P_+ be the collection of all monic irreducible polynomials in A .

For each $t \in \mathbb{C}_\infty$, let $\chi_t : A \rightarrow \mathbb{C}_\infty$ be the canonical \mathbb{F}_q -algebra morphism given by $\theta \mapsto t$. In [10], Pellarin introduces the formal characteristic p valued L -series

$$L(\chi_t, k) = \sum_{a \in A_+} \chi_t(a) a^{-k} = \prod_{v \in P_+} \left(1 - \frac{\chi_t(v)}{v^k} \right)^{-1}.$$

It is easy to see that both the product and series representations converge for $|t|_\infty < |\theta|_\infty$ for all positive integers k . Pellarin's series is a deformation of the characteristic p valued zeta function

$$\zeta(k) := \sum_{a \in A_+} a^{-k}$$

first studied for positive integers k by Carlitz in the 30's and 40's and rediscovered and vastly extended by Goss in the 70's and 80's, see [3, 4] and [8], Chapter

8. In relation to this zeta function Carlitz pioneered machinery which is absolutely basic today to function field arithmetic's analog of classical cyclotomic theory.

Let ι be a fixed choice of $(q-1)$ -th root of $-\theta$. Carlitz discovered the following constant,

$$\tilde{\pi} = \iota\theta \prod_{i=1}^{\infty} \left(1 - \frac{\theta}{\theta^{q^i}}\right)^{-1},$$

and this form is due to Anderson and Thakur in [2]. This constant is algebraic over K_{∞} , and its positive powers $l \equiv 0 \pmod{q-1}$ lie in K_{∞} . Using this number Carlitz showed $\zeta(l)/\tilde{\pi}^l$ is an element of K . This is Carlitz' analog of Euler's classical result for the Riemann zeta function and is now commonly referred to as the *Euler-Carlitz relations* for the Goss-Carlitz zeta function.

Another discovery of Carlitz is an A -module module structure on \mathbb{C}_{∞} which plays the role for function fields that the \mathbb{Z} -module structure of exponentiation on the multiplicative group of the complex numbers plays classically. The *Carlitz module* C is the \mathbb{F}_q -algebra morphism from A into the *twisted polynomial ring* $A\{\tau\}$ determined by sending $\theta \mapsto \tau + \theta\tau^0$, where the elements of $A\{\tau\}$ are subject to the commutation relation $\tau x = x^q\tau$ for all $x \in A$. There is an obvious definition of evaluation for elements x in an A -algebra L given by $\sum a_i \tau^i(x) := \sum a_i x^{q^i}$. Via this definition the A -algebra L obtains a new A -module structure where the twisted polynomial $C(a)$ acts on $x \in L$ by evaluation.

In the 80's Anderson constructed a category of t -modules, containing the Carlitz module, and defined a notion of tensor product for this category. In an important work [2] Anderson and Thakur investigated the tensor powers of the Carlitz module in Anderson's category of t -modules and related the n -th tensor power of the Carlitz module to the value $\zeta(n)$ of the Goss-Carlitz zeta function. This paved the way for results on the transcendence of [13] (and later a classification of algebraic relations among [6]) zeta values at the positive integers.

The following function plays a crucial role in all tensor powers of the Carlitz module. Let

$$\omega(t) := \iota \prod_{i=0}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1}.$$

This product converges for $|t| < |\theta|$, and one observes that $\lim_{t \rightarrow \theta} (\theta - t)\omega(t) = \tilde{\pi}$. Our next result connects all of these stories together, and section 3 is devoted to its proof.

Theorem 1.1 (Pellarin [10], Theorems 1 and 2). *Let k be a positive integer such that $k \equiv 1 \pmod{q-1}$. Then*

$$L(\chi_t, k)\omega(t) = \lambda_k(t)\tilde{\pi}^k$$

for some $\lambda_k(t) \in K(t)$.

Furthermore, when $k = 1$ we have $\lambda_k(t) = 1/(\theta - t)$.

This is clearly a vast generalization of Euler's classical theorem on zeta values. Pellarin discovered this formula using the theory of deformations of vectorial modular forms and this supplied the long desired connection between characteristic p valued L -series and modular forms.

Pellarin has also given an elegant and elementary proof of the formula for $L(\chi_t, 1)$ inspired by Anderson's theory of t -motives, and we reproduce the essential ideas here. Along the way we introduce and prove a certain *fundamental relation* (see Theorem 1.2 just below) relating Pellarin's L -series at $k = 1$ to the Carlitz logarithm. Relations between this function and zeta values are well known, see e.g. [1], [6], [13]. In a forthcoming work of the author and F. Pellarin relations of this type play a fundamental role in computing closed form expressions for Pellarin's L -series at various values of k where one slightly varies the characters involved as follows: Let s be a positive integer, and let $t_i \in \mathbb{C}_\infty$ be such that $|t_i| \leq 1$ for all $i = 1, \dots, s$. For positive integers k we define

$$L(\Pi^s \chi_{t_i}, k) := \sum_{a \in A_+} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{a^k} = \prod_{v \in P_+} \left(1 - \frac{\chi_{t_1}(v) \cdots \chi_{t_s}(v)}{v^k} \right)^{-1}.$$

See Remark 7 in [10] where these series were first defined, and where it is sketched that these series are entire analytic in the variables t_1, \dots, t_s .

The following three objects will appear in most results that follow.

1. Let $b_0(t) = 1$, and for positive integers $i \geq 1$ let $b_i(t) = \prod_{j=0}^{i-1} (t - \theta^{q^j})$.
2. Let $D_0 = 1$, and for positive integers $i \geq 1$ let $D_i = \prod_{j=0}^{i-1} (\theta^{q^i} - \theta^{q^j})$.
3. Let $L_0 = 1$, and for positive integers $i \geq 1$ let $L_i = \prod_{j=1}^i (\theta^{q^j} - \theta)$.

The polynomials $b_i(t)$ are “universal” in the sense that they give rise to the basic objects D_i, L_i (first defined by Carlitz in the 30's) in a simple way. The connection $D_i = b_i(\theta^{q^i})$ is clear, and we will further illuminate it for the L_i when we develop the formalism of the “partial Frobenius.” The following theorem will be proven in Section 4 and provides a concrete example of the above discussion.

Theorem 1.2 (The Generalized Fundamental Relation). *Let $1 \leq r < q$ be a positive integer and $|t_i| \leq 1$ for $i = 1, \dots, r$. Then*

$$L(\Pi^r \chi_{t_i}, 1) = \sum_{i=0}^{\infty} (-1)^i \frac{\prod_{j=1}^r b_i(t_j)}{L_i} \in \mathbb{C}_\infty.$$

As an application of this theorem, we will give explicit representations for all Dirichlet L -series arising from primitive characters $\chi : A \rightarrow \mathbb{F}_q$ in terms of the Carlitz' logarithm and a monic uniformizer of the character's conductor.

Our main result gives an explicit expression in terms of classical Carlitz-Bernoulli numbers and Carlitz factorials (both of which will be defined in the appropriate section) for the rational functions $\lambda_k(t)$ defined in Pellarin's theorem. Our proof is elementary and uses several ingredients: We will use Pellarin's formula for $L(\chi_t, 1)$, and certain expansions of basic functions (such as the character χ_t). Furthermore, the vanishing of Goss' zeta function (which has yet to be defined in this paper) at the negative “even” integers, and the vanishing of the power sums $\sum_{a \in A_+(d)} a^k$ for certain non-negatives integers d, k will be used in a crucial way.

Theorem 1.3 (Main Result). 1. Let k, s be positive integers such that $1 \leq s \leq q$, $k \geq s$ and $k \equiv s \pmod{q-1}$, and let $t_i \in \mathbb{C}_\infty$ be such that $|t_i| \leq 1$ for $i = 1, \dots, s$. Then

$$L(\Pi^s \chi_{t_i}, k) \prod_{i=1}^s \omega(t_i) = \sum_{k_1, \dots, k_s} \left[\zeta \left(k - \sum_{i=1}^s q^{k_i} \right) \prod_{i=1}^s \frac{\tilde{\pi}^{q^{k_i}}}{D_{k_i}(\theta^{q^{k_i}} - t_i)} \right], \quad (1)$$

where the sum is over all combinations of s positive integers k_1, \dots, k_s such that $k - \sum_{i=1}^s q^{k_i} \geq 0$, and ζ is Goss' zeta function.

2. With the above assumptions on k, s, t_i ,

$$L(\Pi^s \chi_{t_i}, k) \prod_{i=1}^s \omega(t_i) = \tilde{\pi}^k \sum_{k_1, \dots, k_s} \left[\frac{BC_{k - \sum_{i=1}^s q^{k_i}}}{\Pi(k - \sum_{i=1}^s q^{k_i})} \prod_{i=1}^s \frac{1}{D_{k_i}(\theta^{q^{k_i}} - t_i)} \right], \quad (2)$$

where the sum is over all combinations of s positive integers k_1, \dots, k_s such that $k - \sum_{i=1}^s q^{k_i} \geq 0$, and the $BC_j \in K$ are the Bernoulli-Carlitz numbers and $\Pi(j) \in A$ is the j -th Carlitz factorial both defined for integers $j \geq 0$.

Remark 1.4. 1. In a private correspondence, for $s > 0$ and $k \equiv s \pmod{q-1}$ F. Pellarin had conjectured the existence of rational functions $\lambda_k(t_1, \dots, t_s) \in K(t_1, \dots, t_s)$ such that

$$L(\Pi^s \chi_{t_i}, k) \prod_{i=1}^s \omega(t_i) = \tilde{\pi}^k \lambda_k(t_1, \dots, t_s).$$

Our main theorem verifies this for $1 \leq s \leq q$, $k \geq s$, and $k \equiv s \pmod{q-1}$. Our method for computing these rational functions λ_k is naturally limited by these restrictions on s and k as can be seen in the proof of the main result.

2. As the right hand sides of the two equations in the statement of our main theorem are finite sums with simple poles agreeing with those of $\prod \omega(t_i)$ we obtain the rigid-analytic continuation of the function $L(\Pi^s \chi_{t_i}, k)$ to an entire function in the variables t_1, \dots, t_s . See [10] Remark 7.
3. As the denominators of the Bernoulli-Carlitz numbers have been given explicitly, see [7], we obtain a complete description of the denominator of $L(\Pi^s \chi_{t_i}, k)$.
4. We also give some discussion of possible generalizations of these formulas and some of the roadblocks to going in those directions following their proofs.

2 First Consequences

In this first section we derive some pleasing first consequences of Pellarin's formula at $k = s = 1$.

As has been observed by Pellarin, if one starts with an element $\lambda \in \overline{\mathbb{F}}_q$ and lets $t = \lambda$ in $L(\chi_t, k)$, then one obtains a characteristic p valued Dirichlet L -function. Indeed, under this substitution χ_t becomes a map $A \rightarrow A/(f)$, where

$f \in A$ is the minimal polynomial of λ . In this way, χ_t may be thought of as a continuous parameterization of characters $\chi : A \rightarrow \overline{\mathbb{F}}_q$.

As an immediate consequence of Pellarin's theorem we recover a product formula, this time over primes of A , for the fundamental period of the Carlitz module $\tilde{\pi}$. M. Papanikolas and his student B. Lutes have shown this by different means by in [9].

Recall that we let P_+ be the set of monic irreducible polynomials of A .

Proposition 2.1. *We have*

$$\tilde{\pi} = \iota\theta \prod_{f \in P_+} \left(1 - \frac{f(0)}{f}\right)^{-1}.$$

Proof. We simply use the product formula for ω first given by Anderson and Thakur

$$\omega(t) = \iota \prod_{i=0}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1},$$

the Euler product for $L(\chi_t, s)$, and Pellarin's formula at $s = 1$. Evaluating at $t = 0$ finishes the proof. \square

Corollary 2.2. *We have*

$$\prod_{f \in P_+} \left(1 - \frac{f(0)}{f}\right) = \prod_{i=1}^{\infty} \left(1 - \frac{\theta}{\theta^{q^i}}\right).$$

Proof. This is immediate. \square

The next consequence of Pellarin's result has some features of an analytic class number formula for the extension $K(((-1)^d f)^{\frac{1}{q-1}})$, where $\mathfrak{p} \in \text{Spec}(A)$ and $f \in \mathfrak{p}$ is a monic generator. Recall that the Galois group $G = \text{Gal}((A/\mathfrak{p})/\mathbb{F}_q)$ is cyclic generated by the Frobenius $\tau : x \mapsto x^q$. We begin with a lemma.

Lemma 2.3. *Let $\lambda \in \overline{\mathbb{F}}_q$, and let $f \in A$ be its minimal polynomial. Suppose f has degree d . Then*

$$\prod_{i=1}^d \omega(\tau^i \lambda)^{q-1} = (-1)^d f.$$

Proof. With the notations above we have $f = \prod_{i=1}^d (\theta - \tau^i \lambda)$. It is a basic property of limits that if $a_n \rightarrow a$ and $b_n \rightarrow b$, then $a_n b_n \rightarrow ab$. With this and

the continuity of the map $x \mapsto x^{q-1}$ we have

$$\begin{aligned}
\prod_{i=1}^d \omega(\tau^i \lambda)^{q-1} &= \prod_{i=1}^d \left(\lim_{n \rightarrow \infty} \iota \prod_{j=0}^n \left(1 - \frac{\tau^i \lambda}{\theta^{q^j}} \right)^{-1} \right)^{q-1} \\
&= \iota^{(q-1)d} \lim_{n \rightarrow \infty} \prod_{i=1}^d \prod_{j=0}^n \left(1 - \frac{\tau^i \lambda}{\theta^{q^j}} \right)^{-(q-1)} \\
&= (-\theta)^d \lim_{n \rightarrow \infty} \prod_{j=0}^n \prod_{i=1}^d \left(1 - \frac{\tau^i \lambda}{\theta^{q^j}} \right)^{-(q-1)} \\
&= (-\theta)^d \lim_{n \rightarrow \infty} \prod_{j=0}^n (f(\theta)/\theta^d)^{-q^j(q-1)} \\
&= (-\theta)^d \lim_{n \rightarrow \infty} (f/\theta^d)^{1-q^{n+1}} \\
&= (-1)^d f.
\end{aligned}$$

The last equality follows since $f/\theta^d = 1 + g$, where $g \in K$ is such that $|g| < 1$, and hence

$$\lim_{n \rightarrow \infty} (f/\theta^d)^{-q^{n+1}} = \lim_{n \rightarrow \infty} (1 + g)^{-q^{n+1}} = \lim_{n \rightarrow \infty} (1 + g^{q^{n+1}})^{-1} = 1.$$

□

Let f, λ be as in the lemma above, and let us fix an isomorphism $A/(f) \cong \mathbb{F}_q(\lambda) \subseteq \mathbb{C}_\infty$ given by $\theta \mapsto \lambda$. We quickly recall some facts about the function field analog of classical cyclotomic extensions. We defined the Carlitz module in the introduction to be the \mathbb{F}_q -algebra morphism C from A into $\mathbb{C}_\infty\{\tau\}$ given by $\theta \mapsto \tau + \theta\tau^0$. For $a \in A$ let us denote the image of a under C by C_a . We examine the polynomial $C_f(x) \in A[x]$, where again we employ the map from the twisted polynomial ring to $A[x]$ given by $\tau(x) = x^q$. Let us write $C[f]$ for the roots of $C_f(x)$. Then it is well known, see [8] or [12], that as an A -module $C[f]$ is isomorphic to $A/(f)$. Furthermore, adjoining the elements of $C[f]$ to K gives an abelian extension whose Galois group G_f is isomorphic to the multiplicative group of $A/(f)$, and in this extension (f) totally ramifies while all other finite primes remain unramified.

Remark 2.4. Let f be as in the lemma above, and let ξ be a root of the polynomial $C_f(x)$. It can be shown, see Rosen [12] Ch. 12, Ex. 1,2, that

$$(-1)^d f = \prod_{a \in A_+(< d)} C_a(\xi)^{q-1},$$

where we define $A_+(< d)$ to be the set of monic polynomials of degree less than d . Thus $\prod \omega(\tau^i \lambda)$ and $\prod C_a(\xi)$ are both $(q-1)$ -th roots of $(-1)^d f$, and hence they differ by an element of \mathbb{F}_q^* . This implies that $\prod \omega(\tau^j \lambda) \in K(C[f])$.

Before stating the next theorem we make a connection with the characters that will appear in its statement and a certain symbol introduced by Carlitz. Via the isomorphism $(A/(f))^* \cong G_f$ we may view the characters $\chi_{\tau^i \lambda}$ and their L -series as associated to the extension $K(C[f])/K$. Now the characters arising

from evaluation at the roots of f , as in the lemma above, are special in that they arise from elements of the Galois group $\text{Gal}((A/(f))/\mathbb{F}_q)$. Indeed, given an element $\sigma_j : x \mapsto x^{q^j} \in \text{Gal}((A/(f))/\mathbb{F}_q)$ we extend this to a character on A via composition with the natural projection $A \rightarrow A/(f) \cong \mathbb{F}_q[\lambda]$. We observe further that for $j = 1, \dots, d$ we have $\chi_{\tau^j \lambda} = \chi_\lambda^{q^j}$. Hence multiplying all of these characters together we obtain

$$\prod_{j=1}^d \chi_{\tau^j \lambda} = \prod_{j=0}^{d-1} \chi_\lambda^{q^j} = \chi_\lambda^{\frac{q^d-1}{q-1}}.$$

We observe that this final character has order $q-1$ and hence takes values in \mathbb{F}_q . In fact, this final character is exactly one of Carlitz' "quadratic symbols" (a/f) which is defined for all $a \in A$ to be the unique element of \mathbb{F}_q such that

$$\left(\frac{a}{f}\right) \equiv a^{\frac{q^d-1}{q-1}} \pmod{f}.$$

One can show that a has a $(q-1)$ -th root modulo f if and only if $(a/f) = 1$. And as we have already sketched there are connections between this symbol and the field $K(C[f])$. There is even a reciprocity law which holds for these characters. M. Rosen gives a delightful account of all of these things in his book [12].

One final reason the characters $\chi_{\tau^j \lambda}$ are special is because once we have fixed an isomorphism from $A/(f) \cong \mathbb{F}_q(\lambda)$ all characters $\chi : A \rightarrow A/(f)$ may be realized as χ_λ^n for some $n = 1, \dots, q^d - 1$. Thus if one were to define new L -values in terms of "digit expansions," as is common in the subject, the L -series associated to these characters would be the base. The reader should keep this in mind when viewing the next result as it may be a possible instance of such constructions. Taking the product over the L -series associated to the special characters discussed above in combination with Pellarin's theorem yields a formula which is reminiscent of classical analytic class number formulas. With the lemma above we are justified in writing $\prod_{j=1}^d \omega(\tau^j \lambda) = ((-1)^d f)^{\frac{1}{q-1}}$.

Theorem 2.5. *Let $\lambda \in \overline{\mathbb{F}_q}$ and let f be its minimal polynomial which we assume is of degree d . Then*

$$\prod_{i=1}^d L(\chi_{\tau^i \lambda}, 1) = \frac{(-1)^d \tilde{\pi}^d}{((-1)^d f)^{\frac{q}{q-1}}}.$$

Proof. From Pellarin's theorem we have

$$L(\chi_{\tau^i \lambda, 1}) \omega(\tau^i \lambda) = \frac{\tilde{\pi}}{\theta - \tau^i \lambda}.$$

Multiplying over $i = 1, \dots, d$ we obtain

$$\prod_{i=1}^d L(\chi_{\tau^i \lambda, 1}) \omega(\tau^i \lambda) = \frac{\tilde{\pi}^d}{f}.$$

Applying the lemma above and doing a little algebra finishes the proof. \square

Remark 2.6. The author speculates that this L -value should be related to the extension $L := K((-1)^d f)^{\frac{1}{q-1}} \subseteq K(C[f])$. Recall again that f is totally ramified in the extension $K(C[f])$, while all other finite primes remain unramified. Thus we have an appearance of a $(q-1)$ -th root of the discriminant of this extension, as well as the appearance of the fundamental period of the Carlitz module $\tilde{\pi}$. It may be possible that the class number, q -regulator, and number of roots of unity (which are all integers) for the ring of integers of L are all sitting in the formula above, but because our L -series take values in a field of characteristic p there is a congruence which is disguising them. For example, it's quite clear that modulo p the number of roots of unity in L is -1 ! There is much left to think about in this direction, but since it would take us afield from the purposes of this note, we leave it for future investigation.

3 Pellarin's Result at 1

In this section we indicate how to derive the formula

$$\omega(t)L(\chi_t, 1) = \frac{\tilde{\pi}}{\theta - t}$$

by elementary means modulo a certain “fundamental identity.” For a similar account see the last section of [10]. We begin by developing the necessary formalism.

Definition 3.1. Let L/K_∞ be a finite extension of fields equipped with the canonical extension of $|\cdot|_\infty$. The *Tate algebra* over L is defined as the L -algebra

$$\mathbb{T}_L = \left\{ \sum_{j \geq 0} a_j t^j : a_j \in L \text{ for all } j, \text{ and } a_j \rightarrow 0 \right\}.$$

Observe that by our assumptions on L it is complete, and hence \mathbb{T}_L is also complete with respect to the norm defined by

$$\|f\| := \max_{i=0}^{\infty} |a_i|,$$

for all $f = \sum a_i t^i \in \mathbb{T}_L$. For now, we fix L and drop the subscript.

We will be interested in understanding certain families of linear endomorphisms of the Tate algebra. Throughout this section we view the universal character χ_t as an embedding of A into \mathbb{T} under the identification $\theta \mapsto t$. Via this embedding one obtains the “trivial action” of A on \mathbb{T} where A acts by left multiplication. More precisely, for all $a \in A$ and $f \in \mathbb{T}$ we define

$$a *_1 f := \chi_t(a)f.$$

Now we may also extend the arithmetic Frobenius $\tau : L \rightarrow L$ sending $x \mapsto x^q$ trivially to an $\mathbb{F}_q[t]$ -linear endomorphism of \mathbb{T} by having it act as follows: For $f = \sum a_i t^i \in \mathbb{T}$ we define

$$\tau * f := \sum a_i^q t^i.$$

The *twisted polynomial ring* $L\{\tau\}$ over L is the non-commutative ring of polynomials in τ satisfying the commutation relation $\tau a = a^q \tau$ for all $a \in L$. With the

action of τ on \mathbb{T} defined above we extend the action of the twisted polynomial ring $L\{\tau\}$ from L to \mathbb{T}_L , and thus we obtain a second action of A on the Tate algebra, this time via the Carlitz module which we now recall.

The *Carlitz module* C is the \mathbb{F}_q -algebra morphism $A \rightarrow A\{\tau\}$ determined by sending $\theta \mapsto \theta\tau^0 + \tau$. We denote the image of the element $a \in A$ under this map by C_a . Via this embedding, any A -algebra J obtains a new A -action by letting C_a , which is a polynomial in τ , act on the elements of J by evaluation.

Thus, to be explicit, the second action of A on the Tate algebra is defined by

$$a *_2 f := C_a * f = \sum C_a(a_i)t^i,$$

where $f = \sum a_i t^i$.

The natural question now arises: Is there a subset of \mathbb{T} on which these two actions agree? The answer is yes, and this subset is a free 1-dimensional $\mathbb{F}_q[t]$ -submodule of \mathbb{T} generated by Anderson and Thakur's function ω . This function originally appeared in [2] as a canonical generator of the 1-dimensional $\mathbb{F}_q[t]$ -module of solutions to the difference equation for $\phi \in \mathbb{T}$

$$\tau\phi = (t - \theta)\phi. \quad (3)$$

Rewriting (3) as $\theta\omega + \tau*\omega = t\omega$, Pellarin observes [10] that this implies $C_a * \omega = \chi_t(a)\omega$ for all $a \in A$ as we have described above. One could say that the function ω “trivializes” the Carlitz action on \mathbb{T} , and in fact ω is closely related to the rigid analytic trivialization of the Carlitz module in the sense of Anderson. In light of the $L\{\tau\}$ action on \mathbb{T} it is convenient to have a special names for the eigenvalues of the powers of τ acting on ω for all positive integers j .

Definition 3.2. Let $b_0(t) = 1$, and for $i \geq 1$, let

$$b_i(t) = \prod_{i=0}^{i-1} (t - \theta^{q^i}).$$

Remark 3.3. 1. One observes immediately that $\tau^j * \omega = b_j(t)\omega$.

2. For $j \geq 1$, one can recover D_j as $b_j(\theta^{q^j})$ and $(-1)^j L_j$ as $(\tau b_j(t))|_{t=\theta}$. Trivially $D_0 = L_0 = b_0(t)$.

3. We will see the polynomials $b_j(t)$ arise in a surprising way in the next section as the coefficients of the “Wagner expansion” of the character χ_t .

Remark 3.4. As another way of looking at things, each element of A gives rise to an $\mathbb{F}_q[t]$ -linear endomorphism of \mathbb{T} via the Carlitz module, and ω is a simultaneous eigenvector for this family of operators. With knowledge of the numerous applications of which the function ω has been a part, it is now very interesting to look for other eigenvectors of the the Carlitz module action of A on \mathbb{T} , or indeed to look for a basis of simultaneous eigenvectors for the commuting family of operators $\{C_a\}_{a \in A}$.

We are almost ready to dive into the proof of Pellarin's result. Before doing so, we must observe that the Carlitz exponential and logarithm act on suitable subspaces of the Tate algebra in a natural way.

For an A -algebra L we define the *ring of twisted power series* $L\{\{\tau\}\}$ to be the set of expressions of the form

$$\sum_{j=0}^{\infty} a_j \tau^j, (a_j \in L \text{ for all } j)$$

with addition and multiplication given by

$$a + b := \sum_i (a_i + b_i) \tau^i \text{ and } ab := \sum_k \left(\sum_{i+j=k} a_i b_j^{q^i} \right) \tau^k$$

respectively, where $a = \sum_i a_i \tau^i$ and $b = \sum_j b_j \tau^j$. Note that this is compatible with the definition of multiplication given above for $A\{\tau\} \subseteq K\{\{\tau\}\}$.

The *Carlitz exponential* is defined to be the series

$$e_C := \sum_{j=0}^{\infty} \frac{\tau^j}{D_j},$$

which is convergent on all of \mathbb{C}_{∞} , with the obvious definition of evaluation:

$$e_C(z) := \sum_{j=0}^{\infty} \frac{z^{q^j}}{D_j}.$$

We also have the composition inverse (see just below for the definition of composition of two such series) of this map called the *Carlitz logarithm*:

$$\log_C := \sum_{j=0}^{\infty} (-1)^j \frac{\tau^j}{L_j},$$

with the analogous definition of evaluation, valid for those $z \in \mathbb{C}_{\infty}$ such that $|z| < \frac{q}{q-1}$.

We also note that if z lies in an extension field L of K , where L is complete, then so do $e_C(z)$ and $\log_C(z)$ whenever they are defined.

For all elements $f = \sum a_i t^i \in \mathbb{T}$ (resp. f such that $\|f\| < q/(q-1)$) we extend e_C (resp. \log_C) to $\mathbb{F}_q[t]$ -linear endomorphisms of \mathbb{T} by

$$e_C * f := \sum e_C(a_i) t^i \text{ (resp. } \log_C * f := \sum \log_C(a_i) t^i \text{)}.$$

With all of the definitions in order, we may now state *The First Fundamental Relation*:

$$L(\chi_t, 1) = \sum_{i=0}^{\infty} (-1)^i \frac{b_i(t)}{L_i}.$$

The proof of a more general result which has this equality as a special case appears as Theorem (1.2) in the next section. Take note of the similarity between the fundamental relation and the expression given above for Carlitz' logarithm. We exploit this now in the proof of Pellarin's Theorem.

Proof of Theorem 1.1. We recall a simple observation of Pellarin [10]. Observe that

$$\omega(t) = \sum_{j=0}^{\infty} e_C \left(\frac{\tilde{\pi}}{\theta^{j+1}} \right) t^j,$$

and that

$$\frac{\tilde{\pi}}{\theta - t} = \sum_{j=0}^{\infty} \frac{\tilde{\pi}}{\theta^{j+1}} t^j.$$

(Note that here we are abusing notation by writing $\tilde{\pi}/(\theta - t)$ for the series $\sum \tilde{\pi}t^i/\theta^{i+1}$, but no confusion should arise as a result of this.) Hence

$$\omega(t) = e_C * [\tilde{\pi}/(\theta - t)]$$

under our definitions. One can check that the largest coefficient of $\omega(t)$ is $e_C(\tilde{\pi}/\theta)$, and that this has absolute value $q^{1/(q-1)}$. Hence the Carlitz logarithm may act on ω , and indeed $\log_C * \omega(t) = \tilde{\pi}/(\theta - t)$. In order to finish the proof of Pellarin's result one just needs to check that the distributive law holds for the Carlitz' logarithm, i.e. that

$$\log_C * \omega(t) = \sum_{j=0}^{\infty} (-1)^j \frac{\tau^j * \omega(t)}{L_j}. \quad (4)$$

Write $\omega(t) = \sum_{i=0}^{\infty} a_i t^i$, then one verifies immediately that $|a_i| = q^{\frac{q}{q-1} - (i+1)}$. Writing (4) using the definitions, the equality to be shown is

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j a_i^{q^j}}{L_i} t^i = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^j a_i^{q^j}}{L_j} t^i. \quad (5)$$

It is enough to show

$$\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left| \frac{(-1)^j a_i^{q^j}}{L_j} t^i \right| < \infty. \quad (6)$$

We estimate:

$$\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left| \frac{(-1)^j a_i^{q^j}}{L_j} t^i \right| \leq \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} q^{\left[\frac{q}{q-1} - (i+1)\right]q^j - \frac{q^{j+1}-q}{q-1}} \quad (7)$$

$$= q^{\frac{q}{q-1}} \sum_{j=0}^{\infty} \frac{1}{q^{q^j} - 1} \quad (8)$$

$$\leq q^{\frac{q}{q-1}} \sum_{j=0}^{\infty} q^{-j} < \infty. \quad (9)$$

Thus (4) holds. Remembering that $\tau^j * \omega(t) = b_j(t)\omega(t)$ and reciting the fundamental relation, Theorem 1.2,

$$L(\chi_t, 1) = \sum_{j=0}^{\infty} (-1)^j \frac{b_j(t)}{L_j}$$

finishes the proof. \square

Remark 3.5. 1. It is interesting to rephrase what we have proved in terms of eigenvectors and eigenvalues for our action of the Carlitz logarithm on the Tate algebra. We have shown that $L(\chi_t, 1)$ is an eigenvalue of the Carlitz logarithm on the vector ω . Explicitly we may write

$$\log_C * \omega = L(\chi_t, 1)\omega.$$

After the proof of our main result, that is Pellarin's whole theorem with the $\lambda_k(t)$ given explicitly, we will give some discussion of possible eigenvectors for the action of Carlitz' logarithm with eigenvalues $L(\chi_t, k)$.

2. Our proof is modeled on that of Pellarin in [10]. He actually proves the following equality of operators on \mathbb{T} :

$$\sum_{a \in A_+} a^{-1} C_a = \log_C.$$

Using this more general point of view Pellarin has results connecting closed form formulas when $1 \leq s < q$ for $L(\chi_t^s, s)$ with the s -th tensor power of the Carlitz module that will appear in a forth coming work of his and the present author.

4 Carlitz Theory and a Fundamental Relation

In this section we indicate how to derive the fundamental relation

$$L(\chi_t, 1) = \sum_{j=0}^{\infty} (-1)^j \frac{b_j(t)}{L_j}$$

using Carlitz' theory of polynomial and series interpolations. The account here is similar to that given in [11], but thanks to the τ formalism of the last section, much less needs to be introduced. Further we produce a small generalization of the fundamental relation in the case that we take instead a product of characters $\prod \chi_{t_i}$ in the definition of Pellarin's L -series. This will be made precise below. As mentioned in the introduction, general relations of this type are right now the key to giving closed form expressions for values of Pellarin's L -series where more general characters are allowed to appear. We continue to view χ_t as the embedding of A into \mathbb{T} sending $\theta \mapsto t$.

We begin with the definitions of several of the basic objects in the theory, the first of which are due to Carlitz.

Definition 4.1. 1. Let $e_0(z) := z$ and for $i \geq 1$ let

$$e_i(z) := \prod_{a \in A(i)} (z - a),$$

where $A(i)$ is the \mathbb{F}_q -vector space of all polynomials in A of degree *strictly less* than i .

2. Recall that we have already defined $D_0 = 1$ and $D_i = \prod_{j=0}^{i-1} (\theta^{q^i} - \theta^{q^j})$ for $i \geq 1$. Now let $i \geq 0$ and let $i = c_r q^r + c_{r-1} q^{r-1} + \cdots + c_1 q + c_0$ be written in its base q expansion. Define the i -th *Carlitz factorial*

$$\Pi(i) := \prod_{j=0}^r D_j^{c_j}.$$

Remark 4.2. It can be shown that the polynomials $e_i(z)$ are \mathbb{F}_q -linear. The essential reason is that their roots form an \mathbb{F}_q -vector space.

The next theorem, originally due to Carlitz, gives a universal expression for the image of the Carlitz module in $\mathbb{C}_{\infty}\{\tau\}$.

Theorem 4.3. For all $z \in \mathbb{C}_\infty$ we define the map

$$z \mapsto \sum_{j=0}^{\infty} \frac{e_j(z)}{D_j} \tau^j \in \mathbb{C}_\infty \{\{\tau\}\}.$$

Then for all $a \in A$ this map has image contained in $A\{\tau\}$ and agrees with the Carlitz module defined earlier (i.e. the map $a \mapsto C_a$).

Proof. See section 3.5 of Goss' book [8]. \square

Remark 4.4. 1. It follows immediately from the definitions for all $z \in \mathbb{C}_\infty$ that $e_j(z)/D_j \rightarrow 0$ as $j \rightarrow \infty$.

2. From here on out we will denote the image of $z \in \mathbb{C}_\infty$ in the map defined in the previous theorem by C_z . Pellarin calls this “an extension of the Carlitz module to all of \mathbb{C}_∞ .”

The next proposition will be used in an essential way in determining the explicit values of Pellarin's L -series at all $k \equiv 1 \pmod{q-1}$.

Proposition 4.5. For all $a \in A$ and all $t \in \mathbb{C}_\infty$ we have

$$\chi_t(a) = \sum_{j=0}^{\infty} b_j(t) \frac{e_j(a)}{D_j}$$

where $\chi_t : a \mapsto a(t) \in \mathbb{T}$ is the quasi-character defined above.

Remark 4.6. Observe that by definition $e_j(a) = 0$ for all j strictly greater than the degree of a and all $a \in A$ so that all sums occurring in the statement above are finite.

Proof. Let $t \in \mathbb{C}_\infty$ be such that $|t| \leq 1$, and let $a \in A$ be fixed. As observed in the last section we have

$$C_a * \omega(t) = \chi_t(a) \omega(t) \text{ and } \tau^j * \omega(t) = b_j(t) \omega(t).$$

But by the last theorem

$$C_a * \omega(t) = \sum \frac{e_j(a)}{D_j} \tau^j * \omega(t).$$

Thus we obtain $\chi_t(a) = \sum \frac{e_j(a)}{D_j} b_j(t)$. Now as a is fixed, we have an equality on the closed unit ball $|t| \leq 1$ of two polynomials. Thus these polynomials must be identically equal. \square

Remark 4.7. 1. It has been shown in [11] that the series

$$\sum_{j=0}^{\infty} b_j(t) \frac{e_j(z)}{D_j}$$

appearing in the previous proposition gives a smooth extension to all of \mathbb{C}_∞ , in the variable z , of the quasi-character χ_t as long as $|t| \leq 1$. Thus the commuting family of operators $\{C_z\}_{z \in \mathbb{C}_\infty}$ has simultaneous eigenvector $\omega(t)$ with continuous family of eigenvalues (which in general are not elements of $\mathbb{F}_q[t]$) given by this

series. This remark is due to F. Pellarin. It should be noted that for $z \in \mathbb{F}_q((\frac{1}{\theta})) \setminus A$ the extension above is no longer “evaluation at t ,” see [11].

2. The familiar reader will recognize the series above as the “Wagner interpolation” for χ_t , and this was the perspective taken in [11]. Indeed, at any finite place \mathfrak{p} of A , for suitable choices of t so that $b_j(t) \rightarrow 0$ in the \mathfrak{p} -adic absolute value, the expression given above is the Wagner interpolation for the character χ_t . Such representations are extremely important in the theory of characteristic p valued measures.

Now we proceed to connect Carlitz’ theory of polynomial interpolations to the representation of χ_t given in the previous proposition. This will lead quickly to the first proof of the fundamental relation. Let B be a commutative A -algebra which is also an integral domain.

Definition 4.8. Let f be a function from A to B . For $d \geq 1$ define the Carlitz polynomial interpolation of f on $A_+(d)$ to be

$$M_d(f)(z) := \sum_{a \in A_+(d)} f(a) \frac{e_d(z-a)}{z-a}.$$

Remark 4.9. 1. Carlitz originally discovered these interpolation polynomials [3], [4] and showed that $(-1)^d (L_d/D_d) M_d(f)(z)$ agrees with $f(z)$ for all $z \in A_+(d)$.

When the function f is \mathbb{F}_q -linear the polynomial $M_d(\chi_t)(z) - M_d(\chi_t)(0)$ is also \mathbb{F}_q -linear and of degree at most $q^d - 1$. Thus it may be expanded in terms of the polynomials $\{e_i(z)\}_{i=0}^{d-1}$. We specialize to $f = \chi_t$ and define the coefficients $\alpha_{d,i}(t)$ by the formula

$$(-1)^d \frac{L_d}{D_d} M_d(\chi_t)(z) = \alpha_{d,-1}(t) + \sum_{i=0}^{d-1} \alpha_{d,i}(t) \frac{e_i(z)}{D_i}. \quad (10)$$

We showed in [11] that $\alpha_{d,i}(t) = b_i(t)$ for $i \geq 0$ and $\alpha_{d,-1}(t) = b_d(t)$, and we now give a different proof.

Proposition 4.10. *The following equality holds in $A[t, z]$:*

$$(-1)^d \frac{L_d}{D_d} M_d(\chi_t)(z) = b_d(t) + \sum_{j=0}^{d-1} b_j(t) \frac{e_j(z)}{D_j}. \quad (11)$$

Proof. The previous proposition implies that for any $a \in A_+(d)$ we have

$$\chi_t(a) = \sum_{j=0}^d b_j(t) \frac{e_j(a)}{D_j}.$$

Recalling that $e_j(a) = D_j$ for all $a \in A_+(d)$ we see that the polynomial $b_d(t) + \sum_{j=0}^{d-1} b_j(t) e_j(z)/D_j \in \mathbb{F}[z, t]$ has degree at most q^{d-1} in z and agrees with $\chi_t(a)$ for all $a \in A_+(d)$. By the first remark above the same is true of $(-1)^d (L_d/D_d) M_d(\chi_t)(z)$ with the exception that its degree in z is at most $q^d - 1$. Nevertheless, since $A_+(d)$ has cardinality q^d , this guarantees that these two polynomials are identically equal finishing the proof of our claim. \square

We begin with a lemma which is similar in spirit to Corollary 3.0.15 of [11].

Lemma 4.11. *Let $1 \leq r < q$ be an integer, and suppose $f_1, \dots, f_r : A \rightarrow B$ are \mathbb{F}_q -linear functions. Let $C_d = (-1)^d D_d / L_d$. Then as elements of $B[z]$ we have*

$$C_d^{r-1} M_d(f_1 \cdots f_r)(z) = \prod_{i=1}^r M_d(f_i)(z).$$

Proof. We have shown in [11] that for \mathbb{F}_q -linear functions f , the polynomial $M_d(f)(z)$ has degree at most q^{d-1} in the variable z . Both the left and right hand sides of the equality to be proved have degree at most $q^d - 1$, and by Carlitz' theory both sides agree under evaluation for all $a \in A_+(d)$ which has q^d elements. Hence as B is an integral domain, these two polynomials must be identically equal. \square

Following Carlitz [3], we now give the first proof of the fundamental relation (and a bit more) through use of the operator M_d . Let \mathbb{T}^r be the Tate algebra in r indeterminates. Explicitly,

$$\mathbb{T}^r := \left\{ \sum a_I t^I : a_I \in \mathbb{C}_\infty \text{ and } a_I \rightarrow 0 \text{ as } |I| \rightarrow \infty \right\},$$

where $I = (i_1, \dots, i_r)$ is a multi-index, and a_I, t^I , and $|I|$ have their usual meanings.

Recall that we have extended the definition of Pellarin's L -series to the formal series:

$$L(\Pi^r \chi_{t_i}, k) := \sum_{a \in A_+} \frac{\prod_{i=1}^r \chi_{t_i}(a)}{a^k},$$

where k is a positive integer. Trivially this series converges for $t_i \in \mathbb{C}_\infty$ with $|t_i| \leq 1$ and for all positive integers k .

Theorem (1.2. The Fundamental Relation). *Let $1 \leq r < q$ be a positive integer. Then as elements of \mathbb{T}^r we have*

$$L(\Pi^r \chi_{t_i}, 1) = \sum_{i=0}^{\infty} (-1)^i \frac{\prod_{j=1}^r b_i(t_j)}{L_i}. \quad (12)$$

Proof. Observe that $e_d(z - a) = e_d(z - \theta^d)$ for all $a \in A_+(d)$ and for all $d \geq 1$. Thus it follows immediately from the lemma just above that

$$\frac{\prod_{i=1}^r M_d(\chi_{t_i})(z)}{e_d(z - \theta^d)} = C_d^{r-1} \sum_{a \in A_+(d)} \frac{\prod_{i=1}^r \chi_{t_i}(a)}{z - a}.$$

After appealing to Remark 4.9.2, evaluating at $z = 0$ and counting finishes the proof. \square

Remark 4.12. With the obvious extensions of F. Pellarin's $\mathbb{F}_q[t]$ -linear τ action on \mathbb{T} to an $\mathbb{F}_q[t_1, \dots, t_r]$ -linear action on \mathbb{T}^r the above result may be interpreted as saying

$$\log_C \left(\prod_{i=1}^r \omega(t_i) \right) = L(\Pi^r \chi_{t_i}, 1) \prod_{i=1}^r \omega(t_i).$$

We recall that $\omega(t_i) = e_C(\tilde{\pi}/(\theta - t_i))$, and observe that it is not clear what $\log_C(\prod_{i=1}^r e_C(\tilde{\pi}/(\theta - t_i)))$ should be exactly; whereas in the case of $r = 1$ such formalism produced Pellarin's result.

Finally we wish to observe that specializing so that $t_i = t$ for all i we can also obtain

$$\log_C(\omega(t)^r) = L(\chi_t^r, 1)\omega(t)^r$$

by means of Proposition 8 (II) in Anderson's paper [1].

Let us give an odd consequence of the proposition above in the case of one character χ_t . In the next corollary we sum Pellarin's L -series by degree. Pulling out our microscope and focusing on a sum of fixed degree $\sum_{a \in A_+(d)} \chi_t(a)a^{-1}$ we begin to see the whole of Pellarin's L -series reemerge as d grows large. Of course, a similar result holds in the case of several characters. We recall ι was the $(q-1)$ -th root of $-\theta$ fixed in the introduction.

Corollary 4.13. *The following limit holds:*

$$\frac{L_d}{\iota \theta^{\frac{q^d-1}{q-1}}} \sum_{a \in A_+(d)} \frac{\chi_t(a)}{a} \rightarrow \frac{(\theta - t)L(\chi_t, 1)}{\tilde{\pi}} \text{ as } d \rightarrow \infty.$$

Proof. The proof of the previous proposition gives for $d \geq 1$:

$$\sum_{a \in A_+(d)} \frac{\chi_t(a)}{a} = \frac{(-1)^d}{L_d} b_d(t).$$

A little elementary algebra gives

$$\frac{L_d}{\iota \theta^{\frac{q^d-1}{q-1}}} \sum_{a \in A_+(d)} \frac{\chi_t(a)}{a} = \iota^{-1} \prod_{i=0}^{d-1} \left(1 - \frac{t}{\theta^{q^i}}\right).$$

Letting d approach infinity yields $1/\omega(t)$ on the right hand side. Recalling Pellarin's result

$$L(\chi_t, 1) = \frac{\tilde{\pi}}{(\theta - t)\omega(t)}$$

finishes the proof. \square

We now specialize so that $t_i = t$ for all i and draw another consequence of the above lemma to give explicit expressions for special values of Goss L -functions at $s = 1$. As we have observed, for any $\lambda \in \overline{\mathbb{F}}_q$, letting $t = \lambda$ in the evaluation character χ_t gives a Dirichlet character $\chi_\lambda : A \rightarrow \overline{\mathbb{F}}_q$. Taking one particular example, we obtain all endomorphisms of the cyclic group \mathbb{F}_q^\times by the power mappings $\{x^m\}_{m=1}^{q-1}$. Hence for $\lambda \in \mathbb{F}_q$ we obtain all Dirichlet characters $\chi : A \rightarrow \mathbb{F}_q$ from the compositions χ_λ^m for $m = 1, \dots, q-1$.

In the next corollary we express the values of Goss' L -series with the family of characters $\{\chi_\lambda^m : \lambda \in \mathbb{F}_q^\times \text{ and } m = 1, \dots, q-1\}$ just described in terms of a multiple of a single Carlitz logarithm evaluated at a $(q-1)$ -th root of a uniformizer for the conductor of the character. For a more in depth study of these Goss L -series at the value 1 and their transcendence properties, see [9].

Corollary 4.14. *Let $\lambda \in \mathbb{F}_q$, and let $(\lambda - \theta)^{\frac{1}{q-1}}$ be a fixed choice of $(q-1)$ -th root of $(\lambda - \theta)$. Then*

$$L(\chi_\lambda^r, 1) = \sum_{a \in A_+} \frac{\chi_\lambda(a)^r}{a} = (\lambda - \theta)^{\frac{-r}{q-1}} \log_C((\lambda - \theta)^{\frac{r}{q-1}}).$$

Proof. Simply substitute $t_i = \lambda$ for $i = 1, \dots, r$ in (12), and reduce. \square

Remark 4.15. 1. As a consistency check, let $r = q - 1$ and $\lambda \in \mathbb{F}_q$. Then from the corollary just above we obtain

$$L(\chi_\lambda^r, 1) = \left(1 - \frac{1}{\theta - \lambda}\right) \zeta(1),$$

where we note that $\chi_\lambda^r = \chi_\lambda^{q-1}$ is the trivial character modulo $(\theta - \lambda)A$, and $1 - 1/(\theta - \lambda)$ is the Euler factor at the prime $(\theta - \lambda)A$. To see this use the functional equation satisfied by the Carlitz logarithm

$$\log_C(\theta z) = \theta \log_C(z) - \log_C(z^q)$$

and the well known fact that $\log_C(1) = \zeta(1)$.

2. The corollary above does not depend on the choice of $(q-1)$ -th root of $\lambda - \theta$, and in light of Lemma 2.3 we may choose $\omega(\lambda)$. Then our corollary becomes

$$\omega(\lambda)^r L(\chi_\lambda^r, 1) = \log_C(\omega(\lambda)^r).$$

This further substantiates the formalism suggested in Remark 4.12.

5 The Main Result

The proof of the main result of this subsection will use knowledge of Goss' extension of the Carlitz zeta function to all of \mathbb{Z} . Further, its proof gives a second derivation of the fundamental relation. We begin with a lemma.

Lemma 5.1. *Let $d \geq 1$ and $k \geq 0$ be integers. Then the power sums*

$$S_d(k) := \sum_{a \in A_+(d)} a^k$$

vanish whenever $k < q^d - 1$.

Proof. See [8] Remark 8.12.1.1. \square

In other words, this lemma says that for fixed k , the power sums vanish when d is sufficiently large. Given this observation we make the next definition.

Definition 5.2. For all integers k we define *Goss' zeta function* by the formula

$$\zeta(k) := \sum_{d=0}^{\infty} \sum_{a \in A_+(d)} a^{-k} \in K_{\infty}.$$

Lemma 5.3. *We have $\zeta(0) = 1$, and for positive integers $k \equiv 0 \pmod{q-1}$ we have $\zeta(-k) = 0$.*

Proof. See [8] Remark 8.12.1.3. □

Remark 5.4. Because of the lemma above we say that Goss' zeta vanishes at the negative “even” integers. Further by the first lemma of this section, we have $\zeta(-k) \in A$ for all positive integers k .

The following theorem is the main result of this paper. We have stated its corollaries in the introduction as Theorem 1.3. Examining the proof in the special case where $s = k = 1$ we obtain a second proof of the fundamental relation. Because the proof is long and technical, we increase the flow of this paper by postponing it until the appendix 6, and here we focus on some immediate corollaries.

Theorem 5.5. *Let s, k be positive integers such that $1 \leq s \leq q$, $k \geq s$ and $k \equiv s \pmod{q-1}$, and let $|t_i| \leq 1$ for $i = 1, \dots, s$. Then the following formula holds:*

$$L(\chi_{t_1} \cdots \chi_{t_s}, k) = \sum_{k_1, \dots, k_s} \left[\zeta \left(k - \sum_{i=1}^s q^{k_i} \right) \prod_{i=1}^s \frac{b_{k_i}(t_i) L(\chi_{t_i}, q^{k_i})}{D_{k_i}} \right], \quad (13)$$

where the sum is over all combinations of s positive integers k_1, \dots, k_s such that $k - \sum_{i=1}^s q^{k_i} \geq 0$.

Remark 5.6. For $s = k$ we obtain a multiplicativity formula for Pellarin's L -series. We have

$$L(\chi_{t_1} \cdots \chi_{t_s}, s) = \prod_{i=1}^s L(\chi_{t_i}, 1).$$

In a forthcoming work of the author and F. Pellarin we will give several different proofs of this result. Specializing so that $t_i = t$ for all i , Pellarin has connected the formula above to the s -th tensor power of the Carlitz' module.

See the remarks following the proof of the next result for a connection with Carlitz' logarithms.

Definition 5.7. We define the l -th Bernoulli-Carlitz numbers by the equation

$$\frac{z}{e_C(z)} = \sum_{l=0}^{\infty} \frac{BC_l}{\Pi(l)} z^l.$$

Remark 5.8. Carlitz used the fact that $e_C(z)$ has kernel $\tilde{\pi}A$ and $1/e_C(z)$ is its logarithmic derivative to show that for $l \equiv 0 \pmod{q-1}$

$$\frac{\zeta(l)}{\tilde{\pi}^l} = \frac{BC_l}{\Pi(l)} \in K.$$

Making this calculation, one observes that the only non-zero coefficients of z appearing in the expansion of $z/e_C(z)$ are those whose index is divisible by $q-1$ (including the index 0).

We now rewrite equation (13) first using Pellarin's result for $L(\chi_t, 1)$, and then using the above expression for $\zeta(k)$ in terms of the Bernoulli-Carlitz numbers.

Theorem (1.3). *i. Let k, s, t_i all be as in the statement of Theorem 5.5, then*

$$L(\chi_{t_1} \cdots \chi_{t_s}, k) \prod_{i=1}^s \omega(t_i) = \sum_{k_1, \dots, k_s} \left[\zeta \left(k - \sum_{i=1}^s q^{k_i} \right) \prod_{i=1}^s \frac{\tilde{\pi} q^{k_i}}{D_{k_i}(\theta q^{k_i} - t_i)} \right], \quad (14)$$

where the sum is over all combinations of s positive integers k_1, \dots, k_s such that $k - \sum_{i=1}^s q^{k_i} \geq 0$.

ii. With the same assumptions on k, s, t_i ,

$$L(\chi_{t_1} \cdots \chi_{t_s}, k) \prod_{i=1}^s \omega(t_i) = \tilde{\pi}^k \sum_{k_1, \dots, k_s} \left[\frac{BC_{k - \sum_{i=1}^s q^{k_i}}}{\Pi(k - \sum_{i=1}^s q^{k_i})} \prod_{i=1}^s \frac{1}{D_{k_i}(\theta q^{k_i} - t_i)} \right], \quad (15)$$

where the sum is over all combinations of s positive integers k_1, \dots, k_s such that $k - \sum_{i=1}^s q^{k_i} \geq 0$.

Proof. We use (13) and Pellarin's formula at $k = 1$. By Pellarin's formula we have

$$(t - \theta)\omega(t)L(\chi_t, 1) = -\tilde{\pi}.$$

Applying the operator τ to both sides j times we obtain

$$b_{j+1}(t)\omega(t)L(\chi_t, q^j) = -\tilde{\pi}q^j.$$

Substituting this into (13) and reducing gives finishes the first claim.

Observing that $D_j = \Pi(q^j)$ and replacing ζ by its Bernoulli-Carlitz representation finishes the second claim. \square

Remark 5.9. Specializing so that $s = 1$ we give an interpretation of equation (14) in terms of Pellarin's operator formalism. Recall that $\log_C * \omega = \tilde{\pi}/(\theta - t)$. Thus (14) becomes

$$\sum_j \frac{\zeta(k - q^j)}{\Pi(q^j)} \tau^j * (\log_C * \omega(t)) = L(\chi_t, k)\omega(t),$$

where the sum is over non-negative integers j such that $k - q^j \geq 0$. Where we had to check that the distributive law holds for our action in the proof of Pellarin's theorem, it holds trivially here as the sum over j is actually finite.

Let $Z_k := \sum_j (\zeta(k - q^j)/\Pi(q^j))\tau^j$ for positive integers $k \equiv 1 \pmod{q-1}$, with the sum over j as above. We obtain the value $L(\chi_t, k)$ as an eigenvalue of the operator $Z_k \circ \log_C$ with eigenvector ω .

Let z be an indeterminate. For $t \in \mathbb{C}_\infty$ such that $|t| \leq 1$ we define the Anderson generating function for the Carlitz module by

$$f_z(t) := \sum_{j=0}^{\infty} e_C \left(\frac{z}{\theta^{j+1}} \right) t^j = \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i(\theta q^i - t)}. \quad (16)$$

Observe that $f_{\tilde{\pi}}(t) = \omega(t)$. Such generating functions are used in general to construct rigid analytic trivializations for Anderson t -motives associated to Drinfeld modules, where in general one replaces the Carlitz exponential function by the exponential for the lattice under consideration, see [5].

Corollary 5.10. *Let s, t_i be as in 5.5, then as formal power series in z*

$$\sum_{a \in A} \frac{\prod_{i=1}^s \chi_{t_i}(a)}{z - a} = \frac{\tilde{\pi}}{e_C(\tilde{\pi}z)} \prod_{i=1}^s \frac{f_{\tilde{\pi}z}(t_i)}{f_{\tilde{\pi}}(t_i)}. \quad (17)$$

Proof. We begin by proving the equality

$$\left(\prod_{i=1}^s f_{\tilde{\pi}}(t_i) \right) \sum_{\substack{k \geq s \\ (q-1) | (k-s)}} L(\Pi^s \chi_{t_i}, k) z^k = \left(\prod_{i=1}^s f_{\tilde{\pi}z}(t_i) \right) \sum_{\substack{n \geq 0 \\ (q-1) | n}} \zeta(n) z^n. \quad (18)$$

On the right hand side of (18) write all of the $f_{\tilde{\pi}z}(t_i)$ in powers of z so that we have

$$\left(\prod_{i=1}^s \sum_{k_i=0}^{\infty} \frac{z^{q^{k_i}}}{D_{k_i}(\theta^{q^{k_i}} - t_i)} \right) \sum_{\substack{n \geq 0 \\ (q-1) | n}} \zeta(n) z^n.$$

Multiplying these power series out, the coefficients of the non-zero powers of z are precisely the right hand sides of (14) as k varies through integers congruent to $s \pmod{q-1}$.

Next observe that

$$\frac{\tilde{\pi}z}{e_C(\tilde{\pi}z)} = \sum_{\substack{n \geq 0 \\ (q-1) | n}} \zeta(n) z^n,$$

and that

$$z \sum_{a \in A} \frac{\prod_{i=1}^s \chi_{t_i}(a)}{z - a} = \sum_{\substack{k \geq s \\ (q-1) | (k-s)}} L(\Pi^s \chi_{t_i}, k) z^k.$$

Making substitutions finishes the proof. \square

Remark 5.11. It would be ideal to prove (17) by methods related to those of Pellarin [10]. Then by equating coefficients, since the right hand side of (17) is entire in the variables t_1, \dots, t_s , we would obtain analytic continuation in t_1, \dots, t_s of Pellarin's L -series for all k appearing on the left-hand-side of this equation as well as explicit expressions as rational functions

As we have done in previous sections, we give a final corollary where we take $\lambda \in \mathbb{F}_q$ and $s = q - 1$ in (15) so that χ_λ^s becomes the trivial character from A to \mathbb{F}_q . Doing so allows us to obtain recursive relations between the Bernoulli-Carlitz numbers.

For positive integers s, n , let $\rho_s(n)$ be the number of s -tuples of non-negative integers (k_1, \dots, k_s) such that $q^{k_1} + \dots + q^{k_s} = n$.

Corollary 5.12. *Let $\lambda \in \mathbb{F}_q$, let $s = q - 1$, and let $k \equiv 0 \pmod{q-1}$. Then*

$$BC_k = \frac{(\theta - \lambda)^k}{1 - (\theta - \lambda)^k} \sum_{n=q-1}^k \frac{\Pi(k)}{\Pi(k-n)\Pi(n)} \frac{\rho_{q-1}(n) BC_{k-n}}{(\theta - \lambda)^{n+1}}. \quad (19)$$

Proof. Use

$$L(\chi_\lambda^{q-1}, k) = \left(1 - \frac{1}{(\theta - \lambda)^k} \right) \zeta(k),$$

Lemma 2.3 and algebra. \square

6 Appendix

In this appendix we provide the proof of our main analytic result:

Theorem (5.5). *Let s, k be positive integers such that $1 \leq s \leq q$, $k \geq s$ and $k \equiv s \pmod{q-1}$, and let $|t_i| \leq 1$ for $i = 1, \dots, s$. Then the following formula holds:*

$$L(\chi_{t_1} \cdots \chi_{t_s}, k) = \sum_{k_1, \dots, k_s} \left[\left(\prod_{i=1}^s \frac{b_{k_i}(t_i) L(\chi_{t_i}, q^{k_i})}{D_{k_i}} \right) \zeta \left(k - \sum_{i=1}^s q^{k_i} \right) \right], \quad (20)$$

where the sum is over all combinations of s positive integers k_1, \dots, k_s such that $k - \sum_{i=1}^s q^{k_i} \geq 0$.

Proof. The proof is rather long. We break it into three lemmas for the reader's convenience. First we recall a few things. As we have seen in section 4, for all $a \in A$ we have

$$\chi_t(a) = \sum_{i=0}^{\deg(a)} b_i(t) \frac{e_i(a)}{D_i}.$$

Further for all $z \in \mathbb{C}_\infty$ we have

$$e_i(z) = \prod_{a \in A(i)} (z - a) = \sum_{j=0}^i \alpha_{i,j} z^{q^j},$$

where $\alpha_{i,j} = (-1)^{i-j} D_i / (D_j L_{i-j}^{q^j})$.

The first lemma uses in a crucial way the well known vanishing of the power sums

$$\sum_{a \in A_+(d')} a^{k'}$$

for positive integers k, d whenever $k \leq q^{d'} - 1$, see Lemma x.1. Its proof gives rise to the condition on s appearing in the main result.

Lemma 6.1 (Condition on s). *Let $1 \leq s \leq q$ and $k \geq s$, then $L(\chi_{t_1}, \dots, \chi_{t_s}, k)$ equals*

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_s=0}^{\infty} \sum_{k_1=0}^{i_1} \cdots \sum_{k_s=0}^{i_s} \left(\prod_{j=1}^s \frac{b_{i_j}(t_j) \alpha_{i_j, k_j}}{D_{i_j}} \right) \zeta(k - (q^{k_1} + \cdots + q^{k_s})). \quad (21)$$

Proof. We begin by replacing χ_{t_i} by its Wagner representation and interchanging sums. The equation

$$\sum_{d=0}^e \sum_{a \in A_+(d)} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{a^k} \quad (22)$$

becomes

$$\sum_{d=0}^e \sum_{i_1=0}^d \cdots \sum_{i_s=0}^d \left(\prod_{j=1}^s \frac{b_{i_j}(t_j)}{D_{i_j}} \right) \sum_{a \in A_+(d)} \left(a^{-k} \prod_{l=1}^s e_{i_l}(a) \right). \quad (23)$$

Next we wish to interchange the sum indexed by the variable d with the sums indexed by the variables i_1, \dots, i_s . It is convenient to observe that due to the vanishing of the polynomials $e_i(b)$ whenever $b \in A(i)$ (i.e. whenever $i > \deg(b)$) we may first increase the upper limits of the sums indexed by i_1, \dots, i_s from d to e without changing anything. Then equation (23) becomes

$$\sum_{i_1=0}^e \cdots \sum_{i_s=0}^e \left(\prod_{j=1}^s \frac{b_{i_j}(t_j)}{D_{i_j}} \right) \sum_{d=0}^e \sum_{a \in A_+(d)} \left(a^{-k} \prod_{l=1}^s e_{i_l}(a) \right). \quad (24)$$

Next we replace the polynomials $e_i(z)$ with their series representations

$$e_{i_j}(z) = \sum_{k_j=0}^{i_j} \alpha_{i_j, k_j} z^{q^{k_j}}$$

and interchange sums. Equation (24) becomes

$$\sum_{i_1=0}^e \cdots \sum_{i_s=0}^e \sum_{k_1=0}^{i_1} \cdots \sum_{k_s=0}^{i_s} \left(\prod_{j=1}^s \frac{b_{i_j}(t_j) \alpha_{i_j, k_j}}{D_{i_j}} \right) \sum_{d=0}^e \sum_{a \in A_+(d)} a^{q^{k_1} + \cdots + q^{k_s} - k}. \quad (25)$$

Next we insert and remove the tail of Goss' zeta function

$$\sum_{d=e+1}^{\infty} \sum_{a \in A_+(d)} a^{q^{k_1} + \cdots + q^{k_s} - k}$$

and (25) becomes

$$\sum_{i_1=0}^e \cdots \sum_{i_s=0}^e \sum_{k_1=0}^{i_1} \cdots \sum_{k_s=0}^{i_s} \left(\prod_{j=1}^s \frac{b_{i_j}(t_j) \alpha_{i_j, k_j}}{D_{i_j}} \right) \zeta(k - q^{k_1} + \cdots + q^{k_s}) - \quad (26)$$

$$\sum_{i_1=0}^e \cdots \sum_{i_s=0}^e \sum_{k_1=0}^{i_1} \cdots \sum_{k_s=0}^{i_s} \left(\prod_{j=1}^s \frac{b_{i_j}(t_j) \alpha_{i_j, k_j}}{D_{i_j}} \right) \sum_{d=e+1}^{\infty} \sum_{a \in A_+(d)} a^{q^{k_1} + \cdots + q^{k_s} - k}. \quad (27)$$

Now we wish to show that the remainder term in (27) vanishes as $e \rightarrow \infty$. In order to do this, we use the well known fact that (see Lemma 5.1) the power sums $\sum_{a \in A_+(d')} a^{k'}$ vanish for non-negative integers k' such that $k' < q^{d'} - 1$ and $d' \geq 1$. We observe that the maximum power of a which can occur in the sum above is $sq^e - k$, and this is less than $q^{e+1} - 1$ since $1 \leq s \leq q$. Hence for all possible d occurring in equation (27) we have $\sum_{i=1}^s q^{k_i} - k < q^d - 1$ (recall that we are also assuming that $k \equiv s \pmod{q-1}$, and what is needed here is simply that $k \geq s$). Hence the upper limits of the sums indexed by k_1, \dots, k_s depend only on k and on i_1, \dots, i_s , but not on e . Thus we will write (27) as

$$\sum_{i_1=0}^e \cdots \sum_{i_s=0}^e \sum_{\hat{k}_1, \dots, \hat{k}_s} \left(\prod_{j=1}^s \frac{b_{i_j}(t_j) \alpha_{i_j, k_j}}{D_{i_j}} \right) \sum_{d=e+1}^{\infty} \sum_{a \in A_+(d)} a^{q^{k_1} + \cdots + q^{k_s} - k}, \quad (28)$$

where the hat indicates only s -tuples (k_1, \dots, k_s) such that $0 \leq k_j \leq i_j$ for $j = 1, \dots, s$ and $\sum_{i=1}^s q^{k_i} - k < 0$ appear.

Now we estimate.

$$\left| \sum_{i_1=0}^e \cdots \sum_{i_s=0}^e \sum_{k_1, \dots, k_s}^{\wedge} \left(\prod_{j=1}^s \frac{b_{i_j}(t_j) \alpha_{i_j, k_j}}{D_{i_j}} \right) \sum_{d=e+1}^{\infty} \sum_{a \in A_+(d)} a^{q^{k_1} + \dots + q^{k_s} - k} \right| \leq \quad (29)$$

$$\sum_{i_1=0}^e \cdots \sum_{i_s=0}^e \sum_{k_1, \dots, k_s}^{\wedge} \prod_{j=1}^s \left| \frac{b_{i_j}(t_j) \alpha_{i_j, k_j}}{D_{i_j}} \right| \sum_{d=e+1}^{\infty} \left| \sum_{a \in A_+(d)} a^{q^{k_1} + \dots + q^{k_s} - k} \right| \leq \quad (30)$$

$$\sum_{i_1=0}^e q^{-q^{i_1}} \cdots \sum_{i_s=0}^e q^{-q^{i_s}} \sum_{k_1, \dots, k_s}^{\wedge} \sum_{j=1}^s q^{-k_j q^{k_j} + \frac{q^{k_j+1}}{q-1}} \sum_{d=e+1}^{\infty} q^{(q^{k_1} + \dots + q^{k_s} - k)d} = \quad (31)$$

$$\sum_{i_1=0}^e q^{-q^{i_1}} \cdots \sum_{i_s=0}^e q^{-q^{i_s}} \sum_{k_1, \dots, k_s}^{\wedge} \sum_{j=1}^s q^{-k_j q^{k_j} + \frac{q^{k_j+1}}{q-1}} \frac{q^{(q^{k_1} + \dots + q^{k_s} - k)(e+1)}}{1 - q^{q^{k_1} + \dots + q^{k_s} - k}} \leq \quad (32)$$

$$q^{-(e+1)} \sum_{i_1=0}^e q^{-q^{i_1}} \cdots \sum_{i_s=0}^e q^{-q^{i_s}} \sum_{k_1, \dots, k_s}^{\wedge} \sum_{j=1}^s \frac{q^{-k_j q^{k_j} + \frac{q^{k_j+1}}{q-1}}}{1 - q^{q^{k_1} + \dots + q^{k_s} - k}} \quad (33)$$

Now the sum indexed by k_1, \dots, k_s is finite and the upper limit depends only on k and i_1, \dots, i_s but not on e for e sufficiently large. We remove the dependence on i_1, \dots, i_s by dropping the condition that $k_j \leq i_j$ for $j = 1, \dots, s$. This has the effect of adding more positive terms to the sum indexed by k_1, \dots, k_s . Thus equation (33) is maximized by

$$q^{-(e+1)} \sum_{k_1, \dots, k_s}^{\wedge} \sum_{j=1}^s \frac{q^{-k_j q^{k_j} + \frac{q^{k_j+1}}{q-1}}}{1 - q^{q^{k_1} + \dots + q^{k_s} - k}} \sum_{i_1=0}^e q^{-q^{i_1}} \cdots \sum_{i_s=0}^e q^{-q^{i_s}}, \quad (34)$$

where the double hat signifies summing over all s -tuples (k_1, \dots, k_s) such that $\sum_{i=1}^s q^{k_i} - k < 0$. The sums indexed by i_1, \dots, i_s are convergent as $e \rightarrow \infty$, the sum indexed by k_1, \dots, k_s is a constant which does not depend on e , and $q^{-(e+1)} \rightarrow 0$ as $e \rightarrow \infty$. Thus we conclude that $L(\chi_{t_1}, \dots, \chi_{t_s}, k)$ equals

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_s=0}^{\infty} \sum_{k_1=0}^{i_1} \cdots \sum_{k_s=0}^{i_s} \left(\prod_{j=1}^s \frac{b_{i_j}(t_j) \alpha_{i_j, k_j}}{D_{i_j}} \right) \zeta(k - (q^{k_1} + \dots + q^{k_s})). \quad (35)$$

□

The second lemma uses the vanishing of Goss' zeta function for negative even integers (Lemma 5.3), i.e. those negative integers divisible by $q - 1$, in a crucial way that is totally analogous to how the vanishing of the power sums was used above. This lemma gives rise to the congruence condition on k appearing in the statement of the main theorem.

Lemma 6.2 (Condition on k). *Let s, k be as in the statement of Theorem 5.5, then $L(\chi_{t_1}, \dots, \chi_{t_s}, k)$ equals*

$$\sum_{k_1, \dots, k_s} \zeta(k - \sum_{i=1}^s q^{k_i}) \left(\sum_{i_1=k_1}^{\infty} \frac{b_{i_1}(t_1) \alpha_{i_1, k_1}}{D_{i_1}} \right) \cdots \left(\sum_{i_s=k_s}^{\infty} \frac{b_{i_s}(t_s) \alpha_{i_s, k_s}}{D_{i_s}} \right), \quad (36)$$

where the sum over k_1, \dots, k_s includes all s -tuples of non-negative integers such that $k - \sum_{i=1}^s q^{k_i} \geq 0$.

Proof. We begin with equation (35), and we observe that by assumption, $k \equiv s \pmod{q-1}$; hence $k - (q^{k_1} + \dots + q^{k_s}) \equiv 0 \pmod{q-1}$. Thus $\zeta(k - (q^{k_1} + \dots + q^{k_s}))$ vanishes as soon as $k - (q^{k_1} + \dots + q^{k_s}) < 0$. This limits the upper bound on the sum indexed by k_1, \dots, k_s and allows for arguments entirely analogous to those above to show that we may interchange sums. \square

We conclude the proof with a little algebraic lemma.

Lemma 6.3. *Let j be a non-negative integer. We have*

$$\sum_{i=j}^{\infty} \frac{b_i(t)}{D_i} \alpha_{i,j} = \frac{b_j(t)}{D_j} L(\chi_t, q^j).$$

Proof.

$$\sum_{i=j}^{\infty} \frac{b_i(t)}{D_i} \alpha_{i,j} = \sum_{i=j}^{\infty} \frac{b_j(t) \tau^j b_{i-j}(t)}{D_i} \frac{(-1)^{i-j} D_i}{D_j L_{i-j}^{q^j}} \quad (37)$$

$$= \frac{b_j(t)}{D_j} \sum_{i=j}^{\infty} (-1)^{i-j} \tau^j \frac{b_{i-j}(t)}{L_{i-j}} \quad (38)$$

$$= \frac{b_j(t)}{D_j} \tau^j \sum_{i=j}^{\infty} (-1)^{i-j} \frac{b_{i-j}(t)}{L_{i-j}} \quad (39)$$

$$= \frac{b_j(t)}{D_j} \tau^j \sum_{i=0}^{\infty} (-1)^i \frac{b_i(t)}{L_i} \quad (40)$$

$$= \frac{b_j(t)}{L_j} \tau^j L(\chi_t, 1) \quad (41)$$

$$(42)$$

In the last two lines we use the fundamental relation. \square

Making the substitution in (36) finishes the proof of the theorem. \square

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